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Wentzel-Kramers-Brillouin Approximation:

The Wentzel-Kramers-Brillouin (WKB) approximation provides an approximate expression of the energy eigenstates. Sometimes it is also called semiclassical approximation. The reason will be clear when we discuss connection to the path integrals.

Lets give an intuitive derivation of the WKB method. For a constant potential we can easily solve the equation for energy eigenstates:

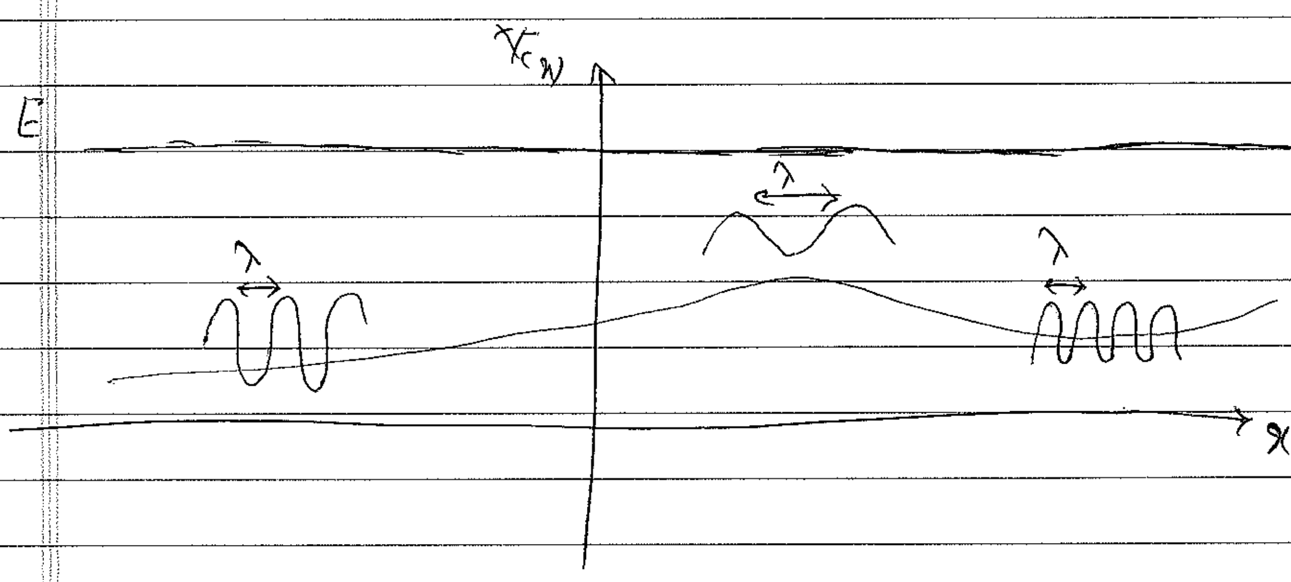
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \Rightarrow \psi \sim \exp\left(\pm \frac{ip}{\hbar}x\right)$$

$$p = \sqrt{2m(E-V_0)}$$

The two solutions correspond to right-moving and left-moving plane waves. The wavelength λ follows:

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{\sqrt{2m(E-V_0)}}$$

For a general potential the solution will not be a plane wave (obviously). However, for a slowly varying potential a wave with varying wavelength might be a good approximation:



We can defined a position-dependent wavelength according to:

$$\lambda(x) = \sqrt{2m(E - V(x))}$$

Note that wavelength represents the distance over which repetition occurs. Therefore position-dependent wavelength makes sense only if the potential changes so slowly that $\delta\lambda$ over one wavelength is much smaller than λ. Otherwise λ will not represent any periodicity in the wave behavior.

Thus:

$$\left|\delta\lambda\,(\text{over 1 wavelength})\right| \simeq \left|\frac{d\lambda(x)}{dx}\right|\lambda << \lambda \Rightarrow \left|\frac{d\lambda(x)}{dx}\right| << 1$$

Therefore the criterion for a slowly varying potential is:

$$\left|\frac{d}{dx}\frac{h}{\sqrt{2m(E-V(x))}}\right| << 1$$

This requires that we be away from the points where $E = V(x)$ (classical bouncing points).

An important observation is that if $h=0$, then we will have $\frac{d\lambda}{dx}=0$. In this case, all potentials are slowly varying. This suggests that "h" can be used as a small parameter whose powers can be used for a series expansion.

To see this quantitatively, lets write the solution to the eigenvalue problem as follows:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_{(x)}\psi = E\psi$$

$$\psi = \exp[i\Phi] \qquad \Phi = \Phi_0 + \hbar\Phi_1 + O(\hbar^2)$$

$$\frac{d^2\psi}{dx^2} = \frac{d^2}{dx^2} e^{i\left(\frac{\Phi_0}{\hbar}+\Phi_1\right)} = \frac{d}{dx}\left[i\left(\frac{\Phi_0'}{\hbar}+\Phi_1'\right) e^{i\left(\frac{\Phi_0}{\hbar}+\Phi_1\right)}\right]$$

$$= +i\left(\frac{\Phi_0''}{\hbar}+\Phi_1''\right) e^{i\left(\frac{\Phi_0}{\hbar}+\Phi_1\right)} \cancel{+} \left(\frac{\Phi_0'}{\hbar}+\Phi_1'\right)^2 e^{i\left(\frac{\Phi_0}{\hbar}+\Phi_1\right)}$$

Then:

$$-i\left(\frac{\Phi_0''}{\hbar}+\Phi_1''\right) + \left(\frac{\Phi_0'^2}{\hbar^2}+\Phi_1'^2+2\frac{\Phi_0'\Phi_1'}{\hbar}\right) = \frac{p^2}{\hbar^2}$$

$$p = \sqrt{2m\left(E-V_{(x)}\right)}$$

Multiplying both sides by $\hbar^2$ and keeping terms up to order $\hbar$, we find:

$$-i(\hbar\Phi_0'') \cancel{+} 2\Phi_0'\Phi_1'\hbar = p^2 - \Phi_0'^2$$

The two sides have different powers of $\hbar$. Hence the only way they can be equal is if they both vanish

This leaves us with two equations:

$$\boxed{\phi_0' = \pm p \Rightarrow \phi_0(x) = \exp\left[\pm \int_{x_0}^{x} p(x')\, dx'\right]}$$

$$\boxed{2\phi_1'\phi_0' = i\phi_0'' \Rightarrow \phi_1' = \frac{ip'}{2p} \Rightarrow \phi_1 = \frac{i \ln p}{2} + \underbrace{C}_{\text{integration constant}}}$$

And:

$$\psi(x) \simeq \exp\left[\frac{i(\phi_0}{\hbar} + \phi_1)\right] = A \exp\left[\frac{\pm i}{\hbar} \int_{x_0}^{x} p(x')\, dx'\right]$$

$$\exp\left(-\frac{\ln p}{2}\right) = \boxed{\frac{A}{\sqrt{p(x)}} \exp\left[\pm \frac{i}{\hbar} \int_{x_0}^{x} p(x')\, dx'\right]}$$

Here $A = \ln C$.

Note that we have the exponential factor that we obtained based on physical reasoning, plus a term $[p(x)]^{-1}$. This is the WKB approximation.